Scattering Theory for Quantum Fields with Indefinite Metric

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Abstract

In this work, we discuss the scattering theory of local, relativistic quantum fields with indefinite metric. Since the results of Haag–Ruelle theory do not carry over to the case of indefinite metric [4], we propose an axiomatic framework for the construction of in- and out- states, such that the LSZ asymptotic condition can be derived from the assumptions. The central mathematical object for this construction is the collection of mixed vacuum expectation values of local, in- and out- fields, called the "form factor functional", which is required to fulfill a Hilbert space structure condition. Given a scattering matrix with polynomial transfer functions, we then construct interpolating, local, relativistic quantum fields with indefinite metric, which fit into the given scattering framework.

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1 Introduction

The Wightman framework of local, relativistic quantum field theory (QFT) turned out to be too narrow for theoretical physicists, who were interested in handling situations involving in particular gauge fields (like in quantum electrodynamics). For several reasons which are intimately connected with the needs of the standard procedure of the perturbative calculation of the scattering matrix (for a detailed discussion, see [32]), the concept of QFT with indefinite metric was introduced, where a probability interpretation is possible only on Hilbert subspaces singled out by a gauge condition in the sense of Gupta [18] and Bleuler [9]. On the other hand, "ghosts", which are quantum fields with the "wrong" connection of spin and statistics, entered the physical scene in connection with the Fadeev–Popov determinant in perturbation theory [15]. As a consequence of Pauli's spin and statistics theorem, such quantum fields can not be realized on a state space with positive metric.

Mathematical foundations for QFT with indefinite metric were laid by several authors, among them Scheibe [29], Yngvason [35], Araki [6], Morchio and Strocchi [26], Mintchev [25] and more recently by G. Hoffmann, see e.g. [22]. The results obtainable from the axioms of indefinite metric QFT in many aspects are less strong than the axiomatic results of positive metric QFT. As the richness of the axiomatic results can be seen as a measure for the difficulty to

construct theories which fulfill such axioms [31], the construction of indefinite metric quantum fields can be expected to be simpler than that of positive metric QFTs.

Up to now, however, the linkage between these mathematical foundations and scattering theory, which in the day to day use of physicists is based on the LSZ reduction formalism [23], remained open, since the only available axiomatic scattering theory (Haag-Ruelle theory[19, 28, 20]) heavily relies on the positivity of Wightman functions. One can even give explicit counter examples of local, relativistic QFTs with indefinite metric [4, 8], such that the LSZ asymptotic condition fails and Haag-Ruelle like scattering amplitudes diverge polynomially in time [4].

A scattering theory for QFTs with indefinite metric which fits well to the LSZ formalism and the mathematically rigorous construction of models of indefinite metric quantum fields (in arbitrary space-time dimension) with nontrivial scattering behavior are the topic of this work, which is organized as follows:

In the second section (and Appendix A) we set up the frame of QFT with indefinite metric and recall some GNS-like results on the representation of $*$-algebras on state spaces with indefinite inner product.

In Section 3 we introduce a set of conditions which is taylored just in the way to imply the LSZ asymptotic condition. The main mathematical object is the collection of mixed expectation values of incoming, local and outgoing fields, called "form factor functional", which is required to fulfill a Hilbert space structure condition (HSSC), cf. [22, 26]. The existence of the form factor functional can be understood as a restriction of the strength of mass-shell singularities in energy-momentum space which rules out the counter examples in [4].

In Section 4 we construct a class of QFTs with indefinite metric and nontrivial scattering behaviour fitting into the frame of Section 3. The main ingredient of this section is a sequence of local, relativistic truncated Wightman functions called the 'structure functions', which have been introduced and studied in [1–5,8,17,18,25]. The non trivial scattering behaviour of the structure functions has been observed in [3, 16, 24]. The class of such QFTs is rich enough to interpolate essentially all scattering matrices with polynomial transfer functions[1]. Some technical proofs can be found in Appendix B.

Section 5 is a supplement to Section 4, in which we discuss the approximation of any set of measurement data for energies below a maximal experimental energy $E_{\max}$ up to an experimental accuracy given by an error tolerance $\epsilon > 0$ with models in the class of Section 4.

2 Quantum fields with indefinite metric

In this section we introduce our notation and we collect some known facts about quantum field theories with indefinite metric following [2, 21, 22, 26].

[1] Schneider, Baumgärtel and Wollenberg constructed a class of weakly local interpolating QFTs with positive metric[7, 30]. These fields however can not be local and are not related to the models we study here.

In order to keep notations simple we study Bosonic, chargeless QFTs[2] over a d dimensional Minkowski space-time $(I\!R^d,\cdot)$ where $x\cdot y = x^0y^0 - \mathbf{x}\cdot\mathbf{y}$ for $x=(x^0,\mathbf{x})=(x^0,x^1,\ldots,x^{d-1}), y=(y^0,\mathbf{y})=(y^0,y^1,\ldots,y^{d-1})\in I\!R^d$. For $x\cdot x$ we will frequently write x^2. The collection of all $k\in I\!R^d$ with $k^2>m^2\geq 0$ and $k^0>0$ ($k^0<0$) is called the forward (backward) mass-cone of mass m and is denoted by the symbol V_m^+ (V_m^-). By $\bar{V}_m^\pm$ we denote the closure of $V_m^\pm$. The (topological) boundary of V_m^+ (V_m^-) is called the forward (backward) mass shell. By $\mathcal{L}$ we denote the full Lorentz group and by $\tilde{\mathcal{P}}_+^\uparrow$ the (covering group of the) orthochronous, proper Poincaré group.

$\mathcal{S}_n$ stands for the the complex valued Schwartz functions over $I\!R^{dn}$and we set $\mathcal{S}_0=\mathbb{C}$. The topology on the spaces $\mathcal{S}_n$ is induced by the Schwartz norms

$$\|f\|_{K,L} = \sup_{\substack{x_1,\ldots,x_n\in I\!R^d\\ 0\leq|\beta_1|,\ldots,|\beta_n|\leq K}} \left|\prod_{l=1}^n (1+|x_l|^2)^{L/2} D^{\beta_1\cdots\beta_n} f(x_1,\ldots,x_n)\right| \tag{1}$$

where $K,L\in I\!N$, $\beta_l=(\beta_l^1,\ldots,\beta_l^{d-1})\in I\!N_0^d, l=1\ldots,n$, are multi indices with $|\beta_l|=\sum_{j=0}^{d-1}\beta_l^j$, $D^{\beta_1\ldots\beta_n}=\prod_{l=1}^n(\partial^{|\beta_l|}/\partial x_l^{\beta_l})$.

The canonical representation α of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{S}_n$ is given by

$$\alpha_{\{\Lambda,a\}}f(x_1,\ldots,x_n)=f(\Lambda^{-1}(x_1-a),\ldots,\Lambda^{-1}(x_n-a)) \tag{2}$$

$\forall\{\Lambda,a\}\in\tilde{P}_+^\uparrow, f\in S_n$.

We normalise the Fourier transform $\mathcal{F}:\mathcal{S}_n\to\mathcal{S}_n$ as follows

$$\mathcal{F}f(k_1,\ldots,k_n)=(2\pi)^{-dn/2}\int_{I\!R^{dn}} e^{-i(x_1\cdot k_1+\cdots+x_n\cdot k_n)}f(x_1,\ldots,x_n)\;dx_1\cdots dx_n \tag{3}$$

$\forall f\in\mathcal{S}_n$. Frequently we will also use the notation $\hat{f}$ instead of $\mathcal{F}f$. For the inverse Fourier transform of f we write $\bar{\mathcal{F}}f$.

Let $\underline{\mathcal{S}}$ be the Borchers' algebra over $\mathcal{S}_1$, namely $\underline{\mathcal{S}}=\bigoplus_{n=0}^\infty\mathcal{S}_n$. $\underline{f}\in\underline{\mathcal{S}}$ can be written in the form $\underline{f}=(f_0,f_1,\ldots,f_j,0,\ldots,0,\ldots)$ with $f_0\in\mathbb{C}$ and $f_n\in\mathcal{S}_n$, $j\in I\!N$.

The addition and multiplication on $\underline{\mathcal{S}}$ are defined as follows:

$$\underline{f}+\underline{h}=(f_0+h_0,f_1+h_1,\ldots) \tag{4}$$

and

$$\begin{aligned}\underline{f}\otimes\underline{h} &= ((\underline{f}\otimes\underline{h})_0,(\underline{f}\otimes\underline{h})_1,\ldots)\\ (\underline{f}\otimes\underline{h})_n &= \sum_{\substack{j,l=0\\ j+l=n}}^\infty f_j\otimes h_l \quad\text{for } n\in I\!N_0\end{aligned} \tag{5}$$

The involution $*$, the Fourier transform $\underline{\mathcal{F}}$ and the representation $\underline{\alpha}$ of $\tilde{\mathcal{P}}_+^\uparrow$ on $\underline{\mathcal{S}}$ are defined through

[2] All results of this article can be generalized to fields with arbitrary parameters, cf. [16].

$$\begin{aligned}\underline{f}^* &= (f_0^*, f_1^*, f_2^* \ldots)\\ \underline{\mathcal{F}f} &= (f_0, \mathcal{F}f_1, \mathcal{F}f_2, \ldots)\\ \underline{\alpha}_{\{\Lambda,a\}}\underline{f} &= (f_0, \alpha_{\{\Lambda,a\}}f_1, \alpha_{\{\Lambda,a\}}f_2, \ldots)\end{aligned} \tag{6}$$

where $f_n^*(x_1, \ldots, x_n) = \overline{f_n(x_n, \ldots, x_1)}$.

We endow $\underline{\mathcal{S}}$ with the strongest topology, such that the relative topology of $\mathcal{S}_n$ in $\underline{\mathcal{S}}$ is the Schwartz topology (direct sum topology). Let $\underline{\mathcal{S}}' = \underline{\mathcal{S}}'(\mathbb{R}^d, \mathbb{C})$ be the topological dual space of $\underline{\mathcal{S}}$. Then $\underline{R} \in \underline{\mathcal{S}}'$ is of the form $\underline{R} = (R_0, R_1, R_2, \ldots)$ with $R_0 \in \mathbb{C}, R_n \in S_n', n \in \mathbb{N}$. Furthermore, any such sequence defines uniquely an element of $\underline{\mathcal{S}}'$. As in the case of $\underline{\mathcal{S}}$, the involution, Fourier transform and representation of $\tilde{\mathcal{P}}_+^\uparrow$ are on $\underline{S}'$ are defined by the corresponding actions on the components $\mathcal{S}_n'$ of $\underline{\mathcal{S}}'$.

Elements of $\underline{\mathcal{S}}'$ are also called Wightman functionals. The tempered distributions $W_n \in \mathcal{S}_n'$ associated to a Wightman functional $\underline{W} \in \underline{\mathcal{S}}'$ are also called (n-point) Wightman functions.

Next we introduce the modified Wightman axioms of Morchio and Strocchi for QFTs in indefinite metric.

Axioms 2.1 A1) Temperedness and normalization: $\underline{W} \in \underline{\mathcal{S}}'$ and $W_0 = 1$.
A2) Poincaré invariance: $\underline{\alpha}_{\{\Lambda,a\}}\underline{W} = \underline{W}\ \forall \{\Lambda, a\} \in \tilde{\mathcal{P}}_+^\uparrow$.
A3) Spectral property: Let $I_{\rm sp}$ be the left ideal in $\underline{\mathcal{S}}$ generated by elements of the form $(0, \ldots, 0, f_n, 0 \ldots)$ with supp$\hat{f}_n \subseteq \{(k_1, \ldots, k_n) \in \mathbb{R}^{dn} : \sum_{l=1}^n k_l \notin \bar{V}_0^+\}$.
Then $I_{\rm sp} \subseteq$ kernel $\underline{W}$.
A4) Locality: Let $I_{\rm loc}$ be the two-sided ideal in $\underline{\mathcal{S}}$ generated by elements of the form $(0, 0, [f_1, h_1], 0, \ldots)$ with supp f_1 and supp h_1 space-like separated.
Then $I_{\rm loc} \subseteq$ kernel $\underline{W}$.
A5) Hilbert space structure condition (HSSC): There exists a Hilbert seminorm $\underline{p}$ on $\underline{\mathcal{S}}$ s.t. $\left|\underline{W}(\underline{f}^* \otimes \underline{g})\right| \leq \underline{p}(\underline{f})\underline{p}(\underline{g}) \forall \underline{f}, \underline{g} \in \underline{\mathcal{S}}$.
A6) Cluster Property: $\lim_{t\to\infty} \underline{W}(\underline{f} \otimes \underline{\alpha}_{\{1,ta\}}\underline{g}) = \underline{W}(\underline{f})\ \underline{W}(\underline{h})\ \forall \underline{f}, \underline{g} \in \underline{\mathcal{S}}, a \in \mathbb{R}^d$ space like (i.e. $a^2 < 0$).
A7) Hermiticity: $\underline{W}^* = \underline{W}$.

All these axioms can be equivalently expressed in terms of Wightman functions in the usual way, cf. [13, 26, 31].

The significance of the Axioms 2.1 can be seen from the following GNS-like construction:

A metric operator $\eta : \mathcal{H} \to \mathcal{H}$ by definition is a self adjoint operator on the separable Hilbert space $(\mathcal{H}, (.,.))$ with $\eta^2 = 1$. Let $\mathcal{D}$ be a dense and linear subspace. We denote the set of (possibly unbounded) Hilbert space operators $A : \mathcal{D} \to \mathcal{D}$ with (restricted) η-adjoint $A^{[*]} = \eta A^* \eta|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ with $\mathsf{O}_\eta(\mathcal{D})$. Clearly, $\mathsf{O}_\eta(\mathcal{D})$ is an unital algebra with involution $[*]$. The canonical topology on $\mathsf{O}_\eta(\mathcal{D})$ is generated by the seminorms $A \to |(\Psi_1, \eta A \Psi_2)|,\ \Psi_1, \Psi_2 \in \mathcal{D}$. We then have the following theorem:

Theorem 2.2 *Let $\underline{W} \in \underline{\mathcal{S}}$ be a Wightman functional which fulfills the Axioms 2.1. Then*
(i) There is a Hilbert space $(\mathcal{H}, (.,.))$ with a distinguished normalized vector $\Psi_0 \in \mathcal{H}$ called the vacuum, a metric operator η with $\eta\Psi_0 = \Psi_0$ inducing a nondegenerate inner product $\langle .,. \rangle = (., \eta .)$ and a continuous $*$-algebra representation $\phi : \underline{\mathcal{S}} \to \mathsf{O}_\eta(\mathcal{D})$ with $\mathcal{D} = \phi(\underline{\mathcal{S}})\Psi_0$ which is connected to the Wightman functional $\underline{W}$ via $\underline{W}(\underline{f}) = \langle \Psi_0, \phi(\underline{f})\Psi_0 \rangle \forall \underline{f} \in \underline{\mathcal{S}}$.
(ii) There is a η-unitary continuous representation $\mathsf{U} : \tilde{\mathcal{P}}_+^\uparrow \to \mathsf{O}_\eta(\mathcal{D})$ ($\mathsf{U}^{[*]} = \mathsf{U}^{-1}$) such that $\mathsf{U}(\Lambda, a)\phi(\underline{f})\mathsf{U}(\Lambda, a)^{-1} = \phi(\underline{\alpha}^{-1}_{\{\Lambda,a\}}\underline{f})$ $\forall \underline{f} \in \underline{\mathcal{S}}, \{\Lambda, a\} \in \tilde{\mathcal{P}}_+^\uparrow$ and Ψ_0 is invariant under the action of U.
(iii) ϕ fulfills the spectral condition $\phi(I_{\rm sp})\Omega = 0$.
(iv) ϕ is a local representation in the sense that $I_{\rm loc} \subseteq \mathrm{kernel}\ \phi$.
(v) For $\Psi_1, \Psi_2 \in \mathcal{D}$ and $a \in I\!R^d$ space like, we get $\lim_{t\to\infty}\langle \Psi_1, \mathsf{U}(1, ta)\Psi_2 \rangle = \langle \Psi_1, \Psi_0 \rangle \langle \Psi_0, \Psi_2 \rangle$.

A quadruple $((\mathcal{H}, \langle .,. \rangle, \Psi_0), \eta, \mathsf{U}, \phi)$ is called a local relativistic QFT with indefinite metric.

Conversely, let $((\mathcal{H}, \langle .,. \rangle, \Psi_0), \eta, \mathsf{U}, \phi)$ be a local relativistic QFT with indefinite metric. Then $\underline{W}(\underline{f}) = \langle \Psi_0, \phi(\underline{f})\Psi_0 \rangle$ $\forall \underline{f} \in \underline{\mathcal{S}}$ defines a Wightman functional $\underline{W} \in \underline{\mathcal{S}}'$ which fulfills the Axioms 2.1.

Proof. See [11, 26]. For the fact that we can chose the metric operator in such a way that $\eta\Psi_0 = \Psi_0$, cf. [22]. Item (v) is just a rephrasing of the cluster property (A6). ∎

It should be mentioned that the pair $(\underline{W}, \underline{p})$ uniquely determines the associated QFT with indefinite metric, but it is believed that in general the Wightman functional $\underline{W}$ admits non equivalent representations as the vacuum expectation value of a QFT with indefinite metric depending on the choice of $\underline{p}$, cf. [6] for a related situation. See however [22] for sufficient conditions s.t. only $\underline{W}$ determines the (maximal) Hilbert space structure.

We want to study sufficient topological conditions on the Wightman functionals which imply the HSSC and therefore the existence of $*$-algebra representations with indefinite metric. To this aim let $\gamma_{K,L}$ be the strongest topology on $\underline{\mathcal{S}}$ s.t. $\forall n \in I\!N$ the restriction of $\gamma_{K,L}$ to $\mathcal{S}_n$ is induced by the norms (1). Let γ be te weakest topology on $\underline{\mathcal{S}}$ generated by all $\gamma_{K,L}$. Then we get e.g. by Theorem 3 of [26]:

Theorem 2.3 *If $\underline{W} \in \underline{\mathcal{S}}'$ fulfills the condition* **(A5')***: $\underline{W}$ is continuous w.r.t. the topology γ, then $\underline{W}$ fulfills the HSSC.*

We note that $\underline{\mathcal{F}}, \bar{\underline{\mathcal{F}}} : \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ are γ-continuous, thus there is no difference between the γ-continuity of $\underline{W}$ and $\hat{\underline{W}}$.

Topological conditions of this kind obviously are "linear" in the sense that they are preserved under linear combinations. The only essentially non-linear condition in the set of Axioms 2.1 thus is the cluster property (A6). It is linearized by an algebraic transformation $\underline{\mathcal{S}}' \ni \underline{W} \mapsto \underline{W}^T \in \underline{\mathcal{S}}'$ known as "truncation". As we shall see, this transformation preserves (A2)-(A4), (A7) and

transforms (A1) into an equivalent linear condition. The crucial observation now is that truncation also preserves the γ-continuity of $\underline{W}$ [2, 21]. Consequently we can translate the modified Wightman axioms 2.1 into a purely linear set of conditions for the truncated Wightman functional. For the technicalities we refer to Appendix A.

3 Construction of asymptotic states

In this section we develop a mathematical framework for scattering in indefinite metric relativistic local QFT. In a certain sense we go in the opposite direction as the axiomatic scattering theory with positive metric [19, 20, 28] where asymptotic fields are being constructed first and the scattering amplitudes are calculated in a second step [20, 23]. Here we postulate the existence of the mixed vacuum expectation values of in- loc- and out- fields and we then construct these fields using the GNS-like procedure of Section 2.

Let $\underline{\mathcal{S}}^{\rm ext}$ be the "extended" Borchers' algebra over the test function space $\mathcal{S}_1^{\rm ext}=\mathcal{S}(\mathbb{R}^d,\mathbb{C}^3)$, which is the space of Schwartz functions with values in $\mathbb{C}^3$. For a =in/loc/out we define $J^a:\mathcal{S}_1\to\mathcal{S}_1^{\rm ext}$ to be the injection of $\mathcal{S}_1$ into the first/second/third component of $\mathcal{S}_1^{\rm ext}$, i.e. $J^{\rm in}f=(f,0,0), J^{\rm loc}f=(0,f,0), J^{\rm out}f=(0,0,f)$, $f\in\mathcal{S}_1$. Then J^a uniquely induces a continuous unital $*$-algebra homomorphism $\underline{J}^a:\underline{\mathcal{S}}\to\underline{\mathcal{S}}^{\rm ext}$ given by $\underline{J}^a=\oplus_{n=0}^\infty J^{a\otimes n}$.

We also define a suitable "projection" $\underline{J}:\underline{\mathcal{S}}^{\rm ext}\to\underline{\mathcal{S}}$ as the unique continuous unital $*$-algebra homomorphism induced by $J:\mathcal{S}_1^{\rm ext}\to\mathcal{S}_1$, $J(f^{\rm in},f^{\rm loc},f^{\rm out})=f^{\rm in}+f^{\rm loc}+f^{\rm out}$.

For simplicity, we only consider the case of only one stable particle mass $m>0$. Let and $\varphi\in C_0^\infty(\mathbb{R},\mathbb{R})$ with support in $(-\epsilon,\epsilon)$ with $0<\epsilon<m^2$ and $\varphi(x)=1$ if $-\epsilon/2<x<\epsilon/2$. We define $\chi^\pm(k)=\theta(\pm k^0)\varphi(k^2-m^2)$ with θ the Heavyside step function and we set

$$\chi_t(a,k)=\begin{cases}\chi^+(k)e^{-i(k^0-\omega)t}+\chi^-(k)e^{-i(k^0+\omega)t} & \text{for } a=\text{in}\\ 1 & \text{for } a=\text{loc}\\ \chi^+(k)e^{i(k^0-\omega)t}+\chi^-(k)e^{i(k^0+\omega)t} & \text{for } a=\text{out}\end{cases} \qquad (7)$$

We then define $\Omega_t:\mathcal{S}_1^{\rm ext}\to\mathcal{S}_1^{\rm ext}$ by

$$\mathcal{F}\Omega_t\bar{\mathcal{F}}=\begin{pmatrix}\chi_t(\text{in},k) & 0 & 0\\ 0 & \chi_t(\text{loc},k) & 0\\ 0 & 0 & \chi_t(\text{out},k)\end{pmatrix}. \qquad (8)$$

Next, we introduce the multi parameter $\underline{t}=(t_1,t_2,\ldots),t_n=(t_n^1,\ldots,t_n^n),t_n^l\in\mathbb{R}$ and we write $\underline{t}\to+\infty$ if $t_n^l\to+\infty$ in any order, i.e. first one t_n^l goes to infinity, then the next etc. . We say that the limit $\underline{t}\to+\infty$ of any given object exists, if it exists for $t_n^l\to+\infty$ in any order and it does not depend on the order. We now define the finite times wave operator $\underline{\Omega}_{\underline{t}}:\underline{\mathcal{S}}^{\rm ext}\to\underline{\mathcal{S}}$ as

$$\underline{\Omega}_{\underline{t}}=\underline{J}\circ\oplus_{n=0}^\infty\Omega_{n,t_n}\ ,\quad \Omega_{0,t_0}=1,\ \ \Omega_{n,t_n}=\otimes_{l=1}^n\Omega_{t_n^l}. \qquad (9)$$

Furthermore, we define the finite times in- and out- wave operators $\underline{\Omega}_{\underline{t}}^{\mathrm{in/out}} : \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ as $\underline{\Omega}_{\underline{t}} \circ \underline{J}^{\mathrm{in/out}}$. Up to changes of the time parameter which do not matter in the limit $\underline{t} \to +\infty$, the wave operators $\underline{\Omega}_{\underline{t}}^{(\mathrm{in/out})}$ are $*$-algebra homomorphisms, as can be easily verified from the definitions.

Definition 3.1 (i) Let $\underline{W} \in \underline{\mathcal{S}}'$ be a Wightman functional s.t. the functionals $\underline{W} \circ \underline{\Omega}_{\underline{t}}$ converge in $\underline{\mathcal{S}}^{\mathrm{ext}'}$ as $\underline{t} \to +\infty$. We then define the form factor functional $\underline{F} \in \underline{\mathcal{S}}^{\mathrm{ext}'}$ associated to $\underline{W}$ as this limit, i.e.

$$\underline{F} = \lim_{\underline{t} \to +\infty} \underline{W} \circ \underline{\Omega}_{\underline{t}}. \tag{10}$$

(ii) The scattering matrix $\underline{S}$ associated to $\underline{W}$ is defined by

$$\begin{aligned} \underline{S}(\underline{f}, \underline{g}) &= \underline{F}(\underline{J}^{\mathrm{in}} \underline{f} \otimes \underline{J}^{\mathrm{out}} \underline{g}) \\ &= \lim_{\underline{t}, \underline{t}' \to +\infty} \underline{W}(\underline{\Omega}_{\underline{t}}^{\mathrm{in}} \underline{f} \otimes \underline{\Omega}_{\underline{t}'}^{\mathrm{out}} \underline{g}) \quad \forall \underline{f}, \underline{g} \in \underline{\mathcal{S}}. \end{aligned} \tag{11}$$

We are now in the position to state a set of conditions which allow a reasonable definition of the scattering matrix, in- and out- fields and states in indefinite metric QFT.

Condition 3.2 Let $\underline{W} \in \underline{\mathcal{S}}'$. We assume that
s1) $\underline{W}$ fulfills Axioms 2.1 and $\underline{W}$ is a theory with a mass gap $m_0 > 0$, i.e. $\underline{W}^T(I_{\mathrm{sp}}^{m_0}) = 0$ with $I_{\mathrm{sp}}^{m_0}$ the *vector space* generated by $(0, \ldots, 0, f_n, 0, \ldots)$ with $\mathrm{supp}\hat{f}_n \subseteq \{(k_1, \ldots, k_n) \in I\!R^{dn} : \exists j,\, 2 \leq j \leq n, \text{ such that} \sum_{l=j}^n k_l \notin \bar{V}_{m_0}^+\}$.
s2) The truncated two point function W_2^T of $\underline{W}$ is of the form

$$\hat{W}_2^T(k_1, k_2) = \left[\delta_m^-(k_1) + \int_{m_0}^{\infty} \delta_\mu^-(k_1)\rho(\mu)d\mu\right] \delta(k_1 + k_2) \tag{12}$$

with ρ a positive polynomially bounded locally integrable density.
s3) The form factor functional $\underline{F}$ associated to $\underline{W}$ exists, is Poincaré invariant and fulfills the Hilbert space structure condition (HSSC).

The following theorem shows that Condition 3.2 just implies the LSZ asymptotic condition.

Theorem 3.3 *We suppose that $\underline{W}$ fulfills the Condition 3.2. Then*
(i) There exists a (in general not local) quantum field theory with indefinite metric $((\mathcal{H}, \langle .,. \rangle, \Psi_0), \eta, \mathsf{U}, \Phi)$ over the Borchers algebra $\underline{\mathcal{S}}^{\mathrm{ext}}$ such that the statements (i)-(iii) of Theorem 2.2 hold.
(ii) There exist relativistic local quantum fields with indefinite metric $\phi^{\mathrm{in/loc/out}} = \Phi \circ \underline{J}^{\mathrm{in/loc/out}}$ over $\underline{\mathcal{S}}$ s.t. $\phi^{\mathrm{in/out}}$ are free fields of mass m (for $d \geq 4$)and $\phi = \phi^{\mathrm{loc}}$ fulfills the LSZ asymptotic condition, namely

$$\lim_{\underline{t} \to +\infty} \phi(\underline{\Omega}_{\underline{t}}^{\mathrm{in/out}} \underline{f}) = \phi^{\mathrm{in/out}}(\underline{f}) \quad \forall \underline{f} \in \underline{\mathcal{S}} \tag{13}$$

where the limit is taken in $\mathsf{O}_\eta(\mathcal{D})$.
(iii) There exist U*-invariant Hilbert spaces* $\mathcal{H}^{\rm in/out} \subseteq \mathcal{H}$ *defined as* $\mathcal{H}^{\rm in/out} = \phi^{\rm in/out}(\underline{\mathcal{S}})\Psi_0$*, s.t. the restriction of* $\langle .,. \rangle$ *to* $\mathcal{H}^{\rm in/out}$ *is positive semidefinite* ($d \geq 4$).

Proof. (i) Except for the spectral property and Hermiticity, this point of the theorem follows immediately from s3) and Theorem 2.2. Concerning the spectral property we note that supp$\mathcal{F}(W_n \circ \Omega_{n,t_n}) \subseteq$ supp $\hat{W}_n$. Thus, supp $\hat{F}_n \subseteq$ supp $\hat{W}_n$. Since $\hat{W}_n$ has the spectral property, which is actually a restriction on the support of $\hat{W}_n$, the spectral property of $\hat{F}_n$ follows. $\Phi(I_{\rm sp})\Omega = \{0\}$ now follows from Theorem 2.2. The Hermiticity follows from the Hermiticity of $\underline{W}$ the fact that $\underline{\Omega_t}$ is a $*$-algebra homomorphism (in the sense given above) and that the limit of Hermitean functionals is Hermitean itself.

(ii) The existence of the fields $\phi^{\rm in/loc/out}$ follows immediately from point (i) of the theorem, namely from the existence of the field Φ. That these fields fulfill the properties of 2.2 for $\phi^{\rm in/out}$ follows from the fact that they are free fields (cf. [31]) and for $\phi^{\rm loc}$ this statement by Theorem 2.2 follows from the assumption s1) on $\underline{W}$.

That $\phi^{\rm in/out}$ for $d \geq 4$are free is a consequence of the fact that the mass gap assumption is fulfilled and thus the truncated Wightman functionals W_n^T fulfill the strong cluster property Theorem XI.110 of [27] Vol. III. Consequently, $\lim_{\underline{t}\to+\infty} \underline{W}^T(\Omega_{\underline{t}}^{\rm in/out}\underline{f}) = 0$ for $f \in \underline{\mathcal{S}}$ with $f_1 = 0, f_2 = 0$ follows from Theorem XI.111 in [27] Vol. III (the negative frequency terms which occur in our framework are just the complex conjugation of some positive frequency term with the same "time direction"). The fact that the locally integrable density $\rho(\mu)d\mu$ does not give a contribution to the two point function of $\phi^{\rm in/out}$ follows from the Riemann lemma, cf. [27] Vol. II (for the details of the argument, see the proof of Proposition 4.7 below).

In order to prove the $\mathsf{O}_\eta(\mathcal{D})$-convergence in Equation (13), we have to show that

$$\lim_{\underline{t}\to+\infty} \left\langle \Phi(\underline{f})\Psi_0, \phi(\underline{\Omega}_{\underline{t}}^{\rm in/out}\underline{g})\Phi(\underline{h})\Psi_0 \right\rangle = \left\langle \Phi(\underline{f})\Psi_0, \phi^{\rm in/out}(\underline{g})\Phi(\underline{h})\Psi_0 \right\rangle$$

holds for all $\underline{g} \in \underline{\mathcal{S}},\ \underline{f}, \underline{h} \in \underline{\mathcal{S}}^{\rm ext}$. Rewriting the left hand side and the right hand side of this formula in terms of the quantum field Φ we can verify it using also s3) by the following calculation

$$\begin{aligned}
& \lim_{\underline{t}\to+\infty} \left\langle \Phi(\underline{f})\Psi_0, \Phi(\underline{J}^{\rm loc}\underline{\Omega}_{\underline{t}}^{\rm in/out}\underline{g})\Phi(\underline{h})\Psi_0 \right\rangle \\
= & \lim_{\underline{t}\to+\infty} \left\langle \Psi_0, \Phi(\underline{f}^* \otimes \underline{J}^{\rm loc}\underline{\Omega}_{\underline{t}}^{\rm in/out}\underline{g} \otimes \underline{h})\Psi_0 \right\rangle \\
= & \lim_{\underline{t}\to+\infty} \underline{F}\left(\underline{f}^* \otimes \underline{J}^{\rm loc}\underline{\Omega}_{\underline{t}}^{\rm in/out}\underline{g} \otimes \underline{h}\right) \\
= & \lim_{\underline{t}\to+\infty} \lim_{\underline{s}_1,\underline{s}_2\to+\infty} \underline{W}\left(\underline{\Omega}_{\underline{s}_1}\underline{f}^* \otimes \underline{\Omega}_{\underline{t}}\underline{J}^{\rm in/out}\underline{g} \otimes \underline{\Omega}_{\underline{s}_2}\underline{h}\right) \\
= & \lim_{\underline{t}\to+\infty} \underline{W}\left(\underline{\Omega}_{\underline{t}}(\underline{f}^* \otimes \underline{J}^{\rm in/out}\underline{g} \otimes \underline{h})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \underline{F}\left(\underline{f}^* \otimes \underline{J}^{\mathrm{in/out}}\underline{g} \otimes \underline{h}\right) \\
&= \left\langle \Psi_0, \Phi(\underline{f}^* \otimes \underline{J}^{\mathrm{in/out}}\underline{g} \otimes \underline{h})\Psi_0 \right\rangle \\
&= \left\langle \Phi(\underline{f})\Psi_0, \Phi(\underline{J}^{\mathrm{in/out}}\underline{g})\Phi(\underline{h})\Psi_0 \right\rangle .
\end{aligned}
$$

(iii) The U-invariance of $\mathcal{H}^{\mathrm{in/out}}$ results from the transformation law $\mathsf{U}(\Lambda,a)\phi^{\mathrm{in/out}}(\underline{f})\mathsf{U}(\Lambda,a)^{-1} = \phi^{\mathrm{in/out}}(\underline{\alpha}^{-1}_{\{\Lambda,a\}}\underline{f})$ and the U-invariance of Ψ_0. The transformation law holds by Theorem 2.2 (ii) and s3). That $\langle .,. \rangle$ is positive semidefinite on $\mathcal{H}^{\mathrm{in/out}}$ follows from the fact that the dense subspaces spaces $\phi^{\mathrm{in/out}}(\underline{\mathcal{S}})\Psi_0$ are also dense subsets of Fock spaces over the one particle space $\mathcal{S}_1$ with positive semidefinite inner product induced by $W_2^{\mathrm{in/out},T}$, cf. [10] p. 288. ■

Here we do not give precise conditions for the existence of the form factor functional, but we refer to the methods of Section 4 and Appendix B where the form factor functional has been constructed for a special class of models. Looking into the details of the proof, one notices that what one really requires to get the existence of this functional is the restriction of mass-shell singularities to singularities of the type $\delta^{\pm}_m(k)$ and $1/(k^2-m^2)$. These are just the singularities occurring in the Feynman propagator. It therefore seems to be reasonable that the form factor functional can be defined in theories where Yang-Feldman equations [34] hold. Since for the physicists' common sense Yang-Feldman equations are an alternative formulation of the LSZ asymptotic condition, to us it seems that our Condition 3.2 does not rule out many cases of physical interest. The assumption of the existence of the form factor functional also can not be dropped from Condition 3.2, since we have to exclude those models from [4] which have too strong mass shell singularities leading to divergent Haag-Ruelle like scattering amplitudes.

Finally in this section we translate the Condition 3.2 into the language of truncated functionals:

Proposition 3.4 *Let $\underline{W} \in \underline{\mathcal{S}}'$ be given and $\underline{W}^T$ be the associated truncated Wightman functional.*
(i) $\underline{F} = \lim_{\underline{t}\to+\infty} \underline{W} \circ \underline{\Omega}_{\underline{t}}$ exists if and only if $\underline{F}^{\tilde{T}} = \lim_{\underline{t}\to+\infty} \underline{W}^T \circ \underline{\Omega}_{\underline{t}}$ exists. In this case $\underline{F}^{\tilde{T}} = \underline{F}^T$ (we may thus omit the tilde in the following).
(ii) $\underline{S}^T(\underline{f},\underline{g}) = \underline{F}^T(\underline{J}^{\mathrm{in}}\underline{f} \otimes \underline{J}^{\mathrm{out}}\underline{g})$ $\forall \underline{f},\underline{g} \in \underline{\mathcal{S}}$.
(iii) Suppose that $\underline{W}^T$ fulfills s1) (transcribed to the language of truncated Wightman functionals according to Proposition A.1) and s2) of Condition 3.2 and furthermore s3T): $\underline{F}^T \in \underline{\mathcal{S}}^{\mathrm{ext}'}$ exists, is Poincaré invariant and γ-continuous. Then the associated Wightman functional fulfills s1)-s3) of Condition 3.2.

Proof. (i) We note that up to the ordering of the time parameter $\underline{t}$ this statement follows from Lemma A.2. But the ordering of $\underline{t}$ does not matter due to the definition of the limit $\underline{t} \to +\infty$.

(ii) This equation follows by application of Lemma A.3 to $\underline{F}$.

(iii) is a corollary to (i), Proposition A.1 and Theorem 2.3. ■

4 An interpolation theorem

In this section we construct a class of quantum fields with indefinite metric which have a well defined scattering behavior in the sense of Theorem 3.3 and which interpolate a certain class of scattering matrices. This is being done by a rather explicit construction of the truncated Wightman functional and the verification of the conditions given in item (iii) of Proposition 3.4. The existence of quantum fields with indefinite metric then follows from Theorem 3.3.

We first recall a well known result of scattering-(S)-matrix theory following [20, 23]: Let us for a moment consider a quantum field as a operator valued distribution $\phi(x)$ (i.e. the restriction of the homomorphism $\phi : \underline{\mathcal{S}} \to \mathsf{O}_\eta(\mathcal{D})$ to $\mathcal{S}_1$). We assume that ϕ fulfills the LSZ-asymptotic condition $\phi\circ\Omega_t^{\rm in/out} \to \phi^{\rm in/out}$ as $t \to \infty$ in an appropriate sense, where the asymptotic fields $\phi^{\rm in/out}$ are free fields of mass m. Let $\hat\phi^{\rm in/out}(k)$ denote the Fourier transform of $\phi^{\rm in/out}(x)$. Then the expectation values of states created by application of the in-fields to the vacuum Ψ_0 with states generated analogously by the out-fields have the following general shape:

$$\begin{aligned} & \left\langle \hat\phi^{\rm in}(k_r)\cdots\hat\phi^{\rm in}(k_1)\Psi_0, \hat\phi^{\rm out}(k_{r+1})\cdots\hat\phi^{\rm out}(k_n)\Psi_0 \right\rangle^T \\ = \quad & 2\pi i \ \underbrace{M_n(-k_r,\ldots,-k_1,k_{r+1},\ldots,k_n)}_{\text{“transfer function”}} \underbrace{\prod_{l=1}^n \delta_m^+(k_l)}_{\text{on}-\text{shell term}} \ \underbrace{\delta(\sum_{l=r+1}^n k_l - \sum_{l=1}^r k_l)}_{\substack{\text{energy}-\text{momentum} \\ \text{conservation term}}} \end{aligned} \tag{14}$$

where $n \geq 3$, $k_l^0 > 0$, $l = 1,\ldots,n$, i.e. all operators $\hat\phi^{\rm in/out}(k_l)$ are creation operators. Since the in- and out- fields fulfill canonical commutation relations this is sufficient to calculate also those expectation values with the condition on the k_l^0 dropped. Here the distribution $M_n(k_1,\ldots,k_n)\delta(\sum_{l=1}^n k_l)$ is given (up to a constant) by the Fourier transform of the time-ordered vacuum expectation values of $\phi(x)$ multiplied by $\prod_{l=1}^n(k_l^2 - m^2)$ and thus is symmetric under permutation of the arguments and Poincaré invariant. By the definition of the scattering matrix in Section 3, we can equivalently write for Equation (14)

$$\begin{aligned} & \hat S^T_{r,n-r}(k_1,\ldots,k_r;k_{r+1},\ldots,k_n) \\ = \quad & 2\pi i \ M_n(k_1,\ldots,k_n) \prod_{l=1}^r \delta_m^-(k_l) \prod_{l=r+1}^n \delta_m^+(k_l) \ \delta(\sum_{l=1}^n k_l) \end{aligned} \tag{15}$$

for $n \geq 3$ and $k_l^0 < 0$ for $l = 1,\ldots,r$ and $k_l^0 > 0$ for $l = r+1,\ldots,n$. Here we used $\hat\phi^{\rm in/out}(k) = \hat\phi^{{\rm in/out}[*]}(-k)$.

Given this general form of the S-matrix, one can ask, whether under some conditions on the transfer functions M_n there exists an interpolating quantum field ϕ s.t. ϕ fulfills the LSZ asymptotic condition and the scattering matrix $\underline{S}$ is determined by Equation (15). In the following we give a (partial) answer to

this question for the case of quantum fields with indefinite metric. First we fix some conditions on the sequence of transfer functions M_n.

Condition 4.1 We assume that $\underline{M} \in \underline{S}'$ fulfills the following conditions:
I1) M_n is symmetric under permutation of arguments and Lorentz invariant (w.r.t. the entire Lorentz group $\mathcal{L}$);
I2) M_n is real, $M_2 = 1$;
I3) M_n is a polynomial;
I4) $\exists L_{\max} \in \mathbb{N}_0$ s. t. $\forall n \in \mathbb{N}$ the degree of $M_n(k_1, \ldots, k_n)$ in any of the arguments $k_1, \ldots, k_n$ is at most $L_{\max}$.

Remark 4.2 The "essentially linear" set of conditions given above of course does not imply unitarity of the scattering matrix, which connects transfer functions of different orders, cf. [10]. Up to now it is not clear, whether in the class of transfer functions described by Condition 4.1 there are exact solutions to the unitarity condition. "Approximate" solutions however are possible due to Proposition 5.1 below.

While the specific properties of the system under consideration are encoded in the transfer functions, we also need an input creating the "axiomatic structure", namely the on-shell terms and the energy-momentum conservation term. In the following we define a sequence of "structure functions[3]" with the required properties.

Definition 4.3 For $n \geq 3$ we define the n-point structure function G_n as the inverse Fourier transform of $\hat{G}_n$ given by

$$\hat{G}_n(k_1, \ldots, k_n) = \left\{ \sum_{j=1}^{n} \prod_{l=1}^{j-1} \delta_m^-(k_l) \frac{1}{k_j^2 - m^2} \prod_{l=j+1}^{n} \delta_m^+(k_l) \right\} \delta(\sum_{l=1}^{n} k_l) \qquad (16)$$

The structure functional $\underline{G} \in \underline{\mathcal{S}}'$ is defined by $G_0 = 0, G_1 = 0$ and $\hat{G}_2$ given by Equation (12).

The structure functions first have been defined in [5] (for $m = 0$), the present form given in Definition 4.3 was obtained in [1]. Further properties of the structure functions are given in [2, 3], see also [16, 17, 24]. The following proposition summarizes the results obtained in these references:

Proposition 4.4 *$\underline{G}$ fulfills all properties of a truncated Wightman functional of a QFT with indefinite metric with a mass gap $m_0 > 0$ (cf. Proposition A.1, Condition 3.2 s1)).*

If $\underline{M}$ is a functional which fulfills Cond. 4.1, then we define the dot-product of the functionals $\underline{M}$ and $\underline{\hat{G}}$ by $(\underline{M} \cdot \underline{\hat{G}})_n = M_n \cdot \hat{G}_n$ where the multiplication on

[3] These functions have nothing to do with the 'structure functions' describing inelastic scattering in the phenomenology of elementary particles.

the right hand side obviously is well defined, since $\hat{G}_n$ is a tempered distribution and M_n is a polynomial.

We now have collected the pieces, which are being put together in the following "interpolation theorem".

Theorem 4.5 *Let $\underline{G}$ be the structure functional (cf. Definition 4.3) and let $\underline{M} \in \underline{\mathcal{S}}'$ fulfill Condition 4.1. Then*
(i) $\underline{\hat{W}}^T = \underline{M} \cdot \underline{\hat{G}}$ fulfills the conditions of Proposition 3.4 (iii).
(ii) The truncated S-matrix (cf. Prop. 3.4 (ii)) is determined by Equation (15).
(iii) In particular, there exists a local, relativistic quantum field ϕ with indefinite metric (see Theorem 2.2) which fulfills the LSZ asymptotic condition Equation (13) w.r.t. free fields $\phi^{\rm in/out}$ of mass m and has scattering behavior determined by Equation (14). The restriction of the indefinite inner product $\langle .,. \rangle$ to the Hilbert spaces $\mathcal{H}^{\rm in/out} = \overline{\phi^{\rm in/out}(\underline{\mathcal{S}})\Psi_0}$ is positive semidefinite.

The rest of this section is devoted to the proof of Theorem 4.5. Obviously, the item (iii) is a straight forward application of (i), (ii), Proposition 3.4 and Theorem 3.3[4]. Therefore, we only have to check the statements (i) and (ii).

Proof of statement (i). *Step 1) Verification of the modified Wightman axioms for $\underline{\hat{W}}^T$ and s1),s2)*: (A1T) holds by $G_0 = 0$ and $\underline{G} \in \underline{\mathcal{S}}'$, cf. Prop. 4.4. Poincaré invariance (A2) follows straightforwardly from the translation invariance of $\underline{G}$ and Lorentz invariance of $\underline{G}$ and $\underline{M}$. The (strong) spectral property (A3) (s1) can be verified by $\mathrm{supp}\hat{W}_n^T = \mathrm{supp}M_n \cdot \hat{G}_n \subseteq \mathrm{supp}\hat{G}_n \subseteq \{(k_1, \ldots, k_n) \in \mathbb{R}^{dn} : \sum_{l=j}^n k_l \in \bar{V}_{m_0}^+ \text{ for } j = 2, \ldots, n\}$ for $n \geq 2$ where the last inclusion holds by Prop. 4.4. Locality (A4) can equivalently be expressed in terms of the (truncated) Wightman functions via $\mathrm{supp}W^T_{n,[,]_j} \subseteq \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn} : (x_j - x_{j+1})^2 \geq 0\}$ for $j = 1, \ldots, n-1$ where $W^T_{n,[,]_j}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = W_n^T(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) - W_n^T(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$. This follows by

$$\mathrm{supp}\ W^T_{n,[,]_j} = \mathrm{supp}\ M_n(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n})G_{n,[,]_j} \subseteq \mathrm{supp}\ G_{n,[,]_j}$$

where in the first step we have made use of the definition of W_n^T and the symmetry of M_n under permutation of the arguments $j, j+1$, and in the second step we used that multiplication by a polynomial in momentum space gives differentiation in position space which is a local operation. Now the assertion follows from the locality of G_n, cf. Prop. 4.4. The proof of (A5') follows from the observation that the $\gamma_{c,r}$-continuity of $\underline{\hat{G}}$ (which holds for some $c, r \in \mathbb{N}$ by Prop. 4.4) implies the $\gamma_{c,r+L_{\max}}$-continuity of $\underline{\hat{W}}^T$ where $L_{\max}$ is given in Condition 4.1 I4). (A6T) follows from the strong spectral property, invariance and locality, cf. Theorem XI.110 of [27] Vol. III. Hermiticity (A7) immediately follows from the Hermiticity of $\underline{G}$ and the fact that

$$\overline{M_n(-k_n, \ldots, -k_1)} = M_n(k_n, \ldots, k_1) = M_n(k_1, \ldots, k_n)$$

[4] By a direct calculation as in the proof of (ii) below one can show that the fields $\phi^{\rm in/out}$ are free fields also for $d = 2, 3$.

where we have also used the real valuedness, reflection invariance and symmetry of M_n. But this is just the relation defining Hermiticity in momentum space. Finally, s2) holds by the definition of $\underline{G}$ and $M_2 = 1$.

Step 2) Calculation of the truncated form factor functional and verification of (A5'), (s3T): We proceed as follows: We define a functional $\underline{F}^G$ and we prove that this is the form factor functional associated to $\underline{G}$. To show this, we require two technical lemmas; their proofs can be found in Appendix B. The rest of the proof of this step is in a similar fashion as the preceding paragraph.

We define the distribution $\Delta_m \in \mathcal{S}_1^{\rm ext}$ by the following formula for the Fourier transform of it's in-, loc- and out- component:

$$\hat{\Delta}_m(a,k) = \begin{cases} -i\pi(\delta_m^+(k) - \delta_m^-(k)) & \text{for } a = \text{in} \\ 1/(k^2 - m^2) & \text{for } a = \text{loc} \\ i\pi(\delta_m^+(k) - \delta_m^-(k)) & \text{for } a = \text{out} \end{cases} \tag{17}$$

Here, as in the definition of the structure functions, the singularity $1/(k^2 - m^2)$ have to be understood in the sense of Cauchy's principal value. We now define the functional $\underline{F}^G$ which turns out to be the form factor functional associated with $\underline{G}$:

Definition 4.6 The functional $\underline{F}^G \in \underline{\mathcal{S}}^{\rm ext'}$ is defined by the following formulae for the Fourier transform of the components $\hat{F}_n^{G(a_1,\ldots,a_n)}$, a_l =in/loc/out, $l = 1,\ldots,n$: $\hat{F}_0^G = 0$, $\hat{F}_1^{G(a_1)}(k_1) = 0$,

$$\hat{F}_2^{G(a_1,a_2)}(k_1,k_2) = \begin{cases} \hat{G}_2(k_1,k_2) & \text{for } a_1 = a_2 = \text{loc} \\ \delta_m^-(k_1)\delta(k_1+k_2) & \text{otherwise} \end{cases} \tag{18}$$

and

$$\hat{F}_n^{G(a_1,\ldots,a_n)}(k_1,\ldots,k_n) = \left\{ \sum_{j=1}^n \prod_{l=1}^{j-1} \delta_m^-(k_l)\hat{\Delta}_m(a_j,k_j) \prod_{l=j+1}^n \delta_m^+(k_l) \right\} \delta(\sum_{l=1}^n k_l). \tag{19}$$

That $\underline{F}^G$ is in $\underline{\mathcal{S}}^{\rm ext'}$, as stated in the Definition 4.6, is contained in the following

Proposition 4.7 *$\underline{F}^G$ is the form factor functional associated to $\underline{G}$. Furthermore, $\underline{F}^G$ is Poincaré invariant and γ-continuous.*

For the proof of this proposition we introduce the test function space $\mathcal{S}_{1,2} = \cap_{L=0}^\infty \overline{\mathcal{S}}_1^{\|\cdot\|_{2,L}}$ (the bar stands for completion) with the topology of the inductive limit. By $\mathcal{S}_{1,2}'$ we denote the topological dual space. It is well-known that $1/(k^2 - m^2)$ as a distribution lies in $\mathcal{S}_{1,2}'$ (since the Cauchy principle value in a neighborhood of the singularity is continuous w.r.t. the C^1-norm) and thus $\hat{\Delta}_m(a,k) \in \mathcal{S}_{1,2}'$ for a = in/loc/out. The following two lemmas contain the analytic part of the proof of Proposition 4.7. For the proof see Appendix B:

Lemma 4.8 $\lim_{t\to+\infty} \frac{\chi_t(a,k)}{(k^2-m^2)} = \hat{\Delta}_m(a,k)$ *holds in* $\mathcal{S}_{1,2}'$ *for* a =in/loc/out.

Lemma 4.9 *For $f \in \mathcal{S}_n$, $n \geq 3, j = 1, \dots, n$, let $g_j : \mathbb{R}^d \to \mathbb{C}$ be defined as*

$$\begin{aligned} g_j(k_j) &= \int_{\mathbb{R}^{d(n-1)}} f(k_1, \dots, k_n) \prod_{l=1}^{j-1} \delta_m^-(k_l) \prod_{l=j+1}^{n} \delta_m^+(k_l) \\ &\times \delta(\sum_{l=1}^n k_l) dk_1 \cdots dk_{j-1} dk_{j+1} \cdots dk_n. \end{aligned} \tag{20}$$

Then $g_j \in \mathcal{S}_{1,2}$ and $\|g_j\|_{2,L} \leq c_L \|f\|_{2,L'}$ for $L \in \mathbb{N}$, $L' = \max\{d, L\}$ and $c_L > 0$ sufficiently large.

Proof of Proposition 4.7 We first note that $\underline{\hat{F}}^G$ is manifestly Poincaré invariant. The γ-continuity of $\underline{F}^G$ can be seen as follows: Let a_l =in/loc/out, $l = 1, \dots, n$ be fixed and $f \in \mathcal{S}_n$, $n \geq 3$. Then by Lemma 4.9 and the fact that $\hat{\Delta}_m(a_j, k_j)$ is continuous w.r.t. $\|.\|_{2,L}$ for $L \geq d + 1$ (with continuity constant $d_L > 0$ sufficiently large) we get the following estimate:

$$\begin{aligned} \left| \hat{F}_n^{G(a_1,\dots,a_n)}(f) \right| &= \left| \sum_{j=1}^n \int_{\mathbb{R}^d} \hat{\Delta}_m(a_j, k_j) g_j(k_j) dk_j \right| \\ &\leq d_L \|g_j\|_{2,L} \leq d_L c_L \|f\|_{2,L}. \end{aligned}$$

Thus, if we choose L sufficiently large s.t. the "continuous part" of $\hat{G}_2$ (which is determined by ρ, cf. Equation 12) is continuous w.r.t. $\|.\|_{0,L}$, we get that $\underline{\hat{F}}^G$ is continuous w.r.t. $\gamma_{2,L}$ and hence w.r.t. γ.

To finish the proof we have to show that for $n \in \mathbb{N}_0$

$$\lim_{t_n^1, \dots, t_n^n \to +\infty} \prod_{l=1}^n \chi_{t_n^l}(a_l, k_l) \hat{G}_n(k_1, \dots, k_n) = \hat{F}_n^{G(a_1,\dots,a_n)}(k_1, \dots, k_n), \tag{21}$$

where $t_n^l \to +\infty$, $l = 1, \dots, n$, in arbitrary order and the limit is being taken in $\mathcal{S}_n'$. For $n = 0, 1$ this holds by definition ($G_0, G_1 = 0$ and $F_0^G, F_G^1 = 0$). Let $n = 2$. For $a_1 = a_2$ =loc there is nothing to prove since $\chi_t(\text{loc}, k) = 1$. Let e.g. a_1 =out and $f \in \mathcal{S}_2$. Then we get for the left hand side of Equation (21) smeared out with f for the case first $t_2^1 \to +\infty$ and then $t_2^2 \to +\infty$

$$\begin{aligned} \dots &= \lim_{t_2^2 \to +\infty} \lim_{t_2^1 \to +\infty} \int_{\mathbb{R}^d} \left[\delta_m^-(k) + \int_{m_0}^\infty \delta_\mu^-(k) \rho(\mu) d\mu \right] \\ &\times e^{i(k^0+\omega)t_2^1} \chi^-(k) \chi_{t_2^2}(a_2, -k) f(k, -k)\ dk \\ &= \int_{\mathbb{R}^d} \delta_m^-(k) f(k, -k)\ dk + \lim_{t_2^2 \to \infty} \lim_{t_2^1 \to +\infty} \int_{m_0}^\infty e^{i(\omega - \omega_\mu) t_2^1} \\ &\times \left[\int_{\mathbb{R}^{d-1}} f((-\mu, \mathbf{k}), (\mu, -\mathbf{k})) \varphi(\mu^2 - m^2) \chi_{t_2^2}(a_2, (-\mu, \mathbf{k})) \frac{d\mathbf{k}}{2\omega_\mu} \right] \rho(\mu) d\mu \end{aligned}$$

Here $\omega_\mu = \sqrt{|\mathbf{k}|^2 + \mu^2}$. We want to show that the limit of the second integral vanishes. To do this, we note that the expression in the brackets [...] defines a

smooth and fast falling (for $\mu \to +\infty$) function in μ and the change of variables $\mu \to \xi = \omega - \omega_\mu$ is smooth (with polynomially bounded determinant) since $m_0 > 0$. Thus, the second integral can be written as the Fourier transform evaluated at t_2^1 of a $L^1(I\!\!R)$-function in the variable ξ (which might depend on t_2^2). By the lemma of Riemann-Lebesgue (cf. Theorem IX.7 [27] Vol. II), the Fourier transform of such a function vanishes at infinity. Thus, the second integral vanishes. If we first take the limit $t_2^2 \to +\infty$ and then $t_2^1 \to \infty$, we can distinguish two cases: If a_2 =loc, the second integral does not depend on t_2^2 and we can thus take the limit $t_2^1 \to +\infty$ as before. If $a_2 \neq$loc we get by an argument which is analogous to the one given above, that the limit $t_2^2 \to +\infty$ of the second integral on the r.h.s. vanishes. This proves Equation (21) for the case $n = 2$.

Let thus $n \geq 3$. Using the fact that $\chi_t(a,k)\delta_m^\pm(k) = \delta_m^\pm(k)$ we get for the left hand side of (21) smeared out with $f \in \mathcal{S}_n$:

$$\ldots = \sum_{j=1}^n \lim_{t_n^j \to +\infty} \int_{I\!\!R^d} \frac{\chi_{t_n^j}(a_j,k_j)}{k_j^2 - m^2} g_j(k_j)\ dk_j .$$

where we have used the notation introduced in Lemma 4.9. Using now that by Lemma 4.9 $g_j \in \mathcal{S}_{1,2}$ we get by Lemma 4.8 for the right hand side of this equation

$$\ldots = \sum_{j=1}^n \int_{I\!\!R^d} \hat{\Delta}_m(a_j,k_j) g_j(k_j)\ dk_j .$$

But this is just the right hand side of Equation (21) smeared out with f. ■

Similar as above, we define the dot-product $\underline{M} \cdot \underline{\hat{F}}^G \in \underline{\mathcal{S}}^{\text{ext}'}$ of $\underline{\hat{F}}^G$ with $\underline{M}$ via $(\underline{M} \cdot \underline{\hat{F}}^G)_n^{(a_1,\ldots,a_n)} = M_n \cdot \hat{F}_n^{G(a_1,\ldots,a_n)}$. We then get from Proposition 4.7 by a simple use of duality and the same arguments as in step 1):

Corollary 4.10 $\underline{F}^T = \bar{\underline{\mathcal{F}}}(\underline{M} \cdot \underline{\hat{F}}^G)$ *exists, is Poincaré invariant and γ-continuous.*

Proof of statement (ii) Let $n \geq 3$, $1 \leq r \leq n-1$, $k_1^0, \ldots, k_r^0 < 0$ and $k_{r+1}^0, \ldots, k_n^0 > 0$. Then by Corollary 4.10

$$\begin{aligned} & \hat{S}^T_{r,n-r}(k_1,\ldots,k_r;k_{r+1},\ldots,k_n) = \hat{F}_n^{T(\text{in},\ldots,\text{in},\text{out},\ldots,\text{out})}(k_1,\ldots,k_n) \\ = \quad & M_n(k_1,\ldots,k_n)\hat{F}_n^{G(\text{in},\ldots,\text{in},\text{out},\ldots,\text{out})}(k_1,\ldots,k_n), \end{aligned}$$

where the "in" is being repeated r times and the "out" $n-r$ times. Inserting (19) into this expression we get

$$\begin{aligned} & M_n(k_1,\ldots,k_n) \left\{ \sum_{j=1}^r \prod_{l=1}^{j-1} \delta_m^-(k_l) \hat{\Delta}_m(\text{in},k_j) \prod_{l=j+1}^n \delta_m^+(k_l) \right. \\ + \quad & \left. \sum_{j=r+1}^n \prod_{l=1}^{j-1} \delta_m^-(k_l) \hat{\Delta}_m(\text{out},k_j) \prod_{l=j+1}^n \delta_m^+(k_l) \right\} \delta(\sum_{l=1}^n k_l). \end{aligned}$$

Using the assumption $k_1^0,\dots,k_r^0 < 0$ and $k_{r+1}^0,\dots,k_n^0 > 0$ for $j = r+1,\dots,n$ we see that in the first sum only the term $j=r$ gives a non vanishing contribution whereas in the second sum all terms vanish except for the term $j=r+1$. Inserting the expression (17) and using $k_r^0<0$ and $k_{r+1}^0>0$ we see that the $\delta_m^+(k_r)$-term in $\hat{\Delta}_m(\text{in},k_r)$ gives no contribution and this is also true for the $\delta_m^-(k_{r+1})$-term in $\hat{\Delta}_m(\text{out},k_{r+1})$. We thus get for the above expression

$$
\begin{aligned}
& M_n(k_1,\dots,k_n)\Bigg\{\prod_{l=1}^{r-1}\delta_m^-(k_l)[i\pi\ \delta_m^-(k_r)]\prod_{l=r+1}^{n}\delta_m^+(k_l)\\
+ & \prod_{l=1}^{r}\delta_m^-(k_l)[i\pi\ \delta_m^+(k_{r+1})]\prod_{l=r+2}^{n}\delta_m^+(k_l)\Bigg\}\delta(\sum_{l=1}^{n}k_l)\\
= & \ 2\pi i\ M_n(k_1,\dots,k_n)\prod_{l=1}^{r}\delta_m^-(k_l)\prod_{l=r+1}^{n}\delta_m^+(k_l)\ \delta(\sum_{l=1}^{n}k_l).
\end{aligned}
$$

This finishes the proof of Theorem 4.5.

5 Approximation of arbitrary scattering amplitudes

Here we want to discuss the approximation of a given ("reference") set of transfer functions (cf. equation (14)) $\underline{R}$ with polynomial transfer functions $\underline{M}$. For $\underline{R}$ we assume full Lorentz invariance (including reflections) and symmetry under permutation of the arguments, which is motivated from the LSZ formalism(see Section 4). Furthermore, we assume that the R_n are continuous, real functions[5]. Since the models of Section 4 have polynomial transfer functions, which grow very fast for large energy arguments and therefore have a somehow 'bad' high energy behaviour, we only consider scattering experiments with maximal energy $E_{\max}>0$, which can be chosen arbitrarily large.

By $Q_n(E_{\max})$, we denote the set of points in energy-momentum space which can be reached by a scattering experiment of maximal energy $E_{\max}$:

$$
\begin{aligned}
\bigcup_{1\le r\le n-1}\Big\{&(k_1,\dots,k_n)\in \mathbb{R}^{dn}: k_l^2=m^2, l=1,\dots,n; k_1^0,\dots,k_r^0<0,\\
&k_{r+1}^0,\dots,k_n^0>0, \sum_{l=r+1}^{n}k_l^0\le E_{\max}, \sum_{l=1}^{n}k_l=0\Big\}.
\end{aligned}
\tag{22}
$$

[5]In general scattering amplitudes are analytic functions on a "cut" neighborhood of the on-shell region and therefore can have discontinuities or singularities on these "cuts", cf. [14, 33]. Therefore, we do not consider $\underline{R}$ as the transfer functions of some "real" theory, but as a set of "measurement data". Then, the requirement of realness can be justified by the fact that only the square modulus of R_n enters in the measurable transition probabilities and continuity can be understood in the sense that R_n was obtained by some continuous interpolation of a discrete set of measurements.

It is easy to verify that for $E_{\max} < \infty$, $Q_n(E_{\max})$ is compact and that $Q_n(E_{\max}) = \emptyset$ for $n > E_{\max}/m$. We say that $\underline{M}$ approximates $\underline{R}$ for energies smaller than $E_{\max}$ up to an error $\epsilon > 0$, if for $n \in \mathbb{N}$ $|M_n(k_1, \ldots, k_n) - R_n(k_1, \ldots, k_n)| < \epsilon$ holds $\forall (k_1, \ldots, k_n) \in Q_n(E_{\max})$. We then get

Proposition 5.1 *Let $\underline{R}$ be a real, fully Lorentz invariant and symmetric functional consisting of continuous functions. For any error parameter $\epsilon > 0$ arbitrarily small and any energy cut-off parameter $E_{\max} > 0$, there exists a functional $\underline{M}$ which fulfills the Conditions 4.1 and which approximates $\underline{R}$ for energies smaller than $E_{\max}$ up to an error ϵ (in the sense given above).*

In particular, there exists a QFT with indefinite metric in the class of QFTs given in Theorem 4.5 with scattering behavior which for energies smaller than $E_{\max}$ differs from the data $\underline{R}$ at most by an error ϵ.

We start the proof with a technical lemma:

Lemma 5.2 *Let $R_n : (\bar{V}^+_{m_0} \cup \bar{V}^-_{m_0}) \times \mathbb{R}^{d(n-1)} \to \mathbb{R}$ be continuous and invariant under the full Lorentz group $\mathcal{L}$. Then there exists a continuous function $V_n : \mathbb{R}^{n(n+1)/2} \to \mathbb{R}$ s.t.*

$$R_n(k_1, \ldots, k_n) = V_n(k_1^2, k_1 \cdot k_2, k_2^2, \ldots, k_1 \cdot k_n, k_2 \cdot k_n, \ldots, k_n^2).$$

Sketch of the Proof. Let $\pi : (\bar{V}^+_{m_0} \cup \bar{V}^-_{m_0}) \times \mathbb{R}^{d(n-1)}/\mathcal{L} \to \mathbb{R}^{n(n+1)/2}$ be defined by $\mathcal{L}\bar{k} = \mathcal{L}(k_1, \ldots, k_n) \to (k_1^2, k_1 \cdot k_2, k_2^2, \ldots, k_1 \cdot k_n, k_2 \cdot k_n, \ldots, k_n^2) = (q_{1,1}, \ldots, q_{n,n}) = \bar{q}$. We want to define V_n on the image of π as $R_n \circ \pi^{-1}$. Hence we have to show that $\bar{k}, \bar{k}' \in \pi^{-1}(\bar{q})$ are in the same orbit of $\mathcal{L}$ in $\mathbb{R}^{dn}$.

First, we can apply a Lorentz boost (possibly in connection with time reflection) which maps k_1 (k_1') to $(\sqrt{k_1^2}, \mathbf{0})$. Then, in this new frame of reference the zero components of $k_l, l = 2, \ldots, n$ are given by $k_l \cdot k_1/\sqrt{k_1^2}$ (for k_l' we proceed analogously). Since the zero components are known, also scalar products of the $\mathbf{k}_l$ $(\mathbf{k}_l')$ are known in this new frame which fixes distances of 'points' from the origin and 'angles' of the 'rigid body' spanned by the $\mathbf{k}_l$ $(\mathbf{k}_l')$ in $\mathbb{R}^{d-1}$. But then there is a orthogonal transformation on $\mathbb{R}^{d-1}$ moving the 'rigid body' spanned by $\mathbf{k}_l$ onto the one spanned by the $\mathbf{k}_l'$. Hence $\bar{k}$ and $\bar{k}'$ are in the same orbit.

Furthermore, the mapping V_n is continuous on the set Ran π. This follows from the fact that one can construct a reference vector $\bar{r}(\bar{q}) \in \mathbb{R}^{dn}$ corresponding to fixing the zero component and a 'standard orientation' for the 'rigid body' which depends smoothly of $\bar{q}$. Thus, for $\bar{q}_n \to \bar{q}'$ in Ran π we get $\bar{r}_n \to \bar{r}'$ and thus $V_n(\bar{q}_n) = R_n(\bar{r}_n) \to R_n(\bar{r}') = V_n(\bar{q}')$. Since Ran π is closed in $\mathbb{R}^{n(n+1)/2}$, there exists a continuous extension of V_n to $\mathbb{R}^{n(n+1)/2}$. ■

Proof of Proposition 5.1 We use the same notations as in the proof of Lemma 5.2. Note that $\pi(Q_n(E_{\max}))$ is compact since π is continuous and $Q_n(E_{\max})$ is compact. Thus, for $\epsilon > 0$ by the Stone-Weierstrass theorem there exists a polynomial p_n such that $|p_n(\bar{q}) - V_n(\bar{q})| < \epsilon$ $\forall \bar{q} \in \pi(Q_n(E_{\max}))$. Let thus $M_n(\underline{k}) = p_n(\pi(\bar{k}))$, then $|M_n(\bar{k}) - R_n(\bar{k})| < \epsilon$ $\forall \bar{k} \in Q_n(E_{\max})$. Furthermore, there is no problem to assume that M_n is real and symmetric under exchange

of variables, since if this is not the case we can replace M_n with $\mathrm{Re}M_n$ and symmetrize without changing the approximation properties.

By construction M_n is invariant under the full Lorentz group. It remains to show that the uniform bound in the degree of $M_n(k_1, \ldots, k_n)$ can be obtained. But this follows from $Q_n(E_{\max}) = \emptyset$ for $n > E_{\max}/m$, which means that we can chose $\{M_n\}_{n>E_{\max}/n}$ as arbitrary real, symmetric and Lorentz invariant polynomials with uniform bound. ■

By Proposition 5.1, there is no 'falsification' based on scattering experiments for the statement that the "true" theory explaining a set of measurements $\underline{R}$ is in the class of models given in Theorem 4.5 (note that $\langle ., . \rangle$ is positive semidefinite on the asymptotic states, thus there is no problem with the probability interpretation of such experiments). Of course, we do not consider this as a serious physical statement. Instead, we think that this result emphasizes the importance of structural aspects (as e.g. a "good" high energy behavior, "exact" unitarity), which might go beyond an explicit and exact measurability.

A Truncation of (bi-) linear functionals on Borchers' algebra

We introduce the following notation: Let $\lambda_l = (\lambda_l^1, \ldots, \lambda_l^j) \subseteq (1, \ldots, n)$ where the inclusion means that λ_l is a subset of $\{1, \ldots, n\}$ and the natural order of $(1, \ldots, n)$ is preserved. Let $\mathcal{P}(1, \ldots, n)$ denote the collection of all partitions of $(1, \ldots, n)$ into disjoint sets λ_l, i.e. for $\lambda \in \mathcal{P}(1, \ldots, n)$ we have $\lambda = \{\lambda_1, \ldots, \lambda_r\}$ for some r where $\lambda_l \subseteq (1, \ldots, n)$, $\lambda_l \cap \lambda_{l'} = \emptyset$ for $l \neq l'$ and $\cup_{l=1}^r \lambda_l = \{1, \ldots, n\}$. Given a Wightman functional $\underline{W} \in \underline{\mathcal{S}}'$ and $\lambda_l = (\lambda_l^1, \ldots, \lambda_l^j)$, we set $W(\lambda_l) = W_j(x_{\lambda_l^1}, \ldots, x_{\lambda_l^j})$.

With this definition at hand we can recursively define the truncated Wightman functional $\underline{W}^T \in \underline{\mathcal{S}}'$ associated to $\underline{W} \in \underline{\mathcal{S}}'$ via $W_0^T = 0$ and

$$W(1, \ldots, n) = \sum_{\lambda \in \mathcal{P}(1, \ldots, n)} \prod_{l=1}^{|\lambda|} W^T(\lambda_l) \ , n \in \mathbb{N}, \tag{23}$$

where $|\lambda|$ is the number of sets λ_l in λ. We have the following proposition on the properties of $\underline{W}^T$:

Proposition A.1 *$\underline{W}$ fulfills the axioms 2.1 (A1)-(A4),(A5'),(A6) and (A7) if and only if $\underline{W}^T$ fulfills (A1T): $W_0 = 0, \underline{W}^T \in \underline{\mathcal{S}}'$, (A2)-(A4), (A5'),(A7) and (A6T): $\lim_{t \to \infty} \underline{W}^T(\underline{f} \otimes \underline{\alpha}_{\{1,ta\}} \underline{g}) = 0$ for $a \in \mathbb{R}^d$ space like and $\underline{f}, \underline{g} \in \underline{S}$ with $f_0 = g_0 = 0$.*

Proof. The equivalence of (A1)/(A2)-(A4)/(A7) for $\underline{W} \Leftrightarrow$ (A1T)/(A2)-(A4)/(A7) for $\underline{W}^T$ can be found e.g. in [10] pp. 492-493. (A6) for $\underline{W} \Leftrightarrow$ (A6T) for $\underline{W}^T$ is well-known, for a detailed proof cf. [1] section 4. (A5') for $\underline{W} \Leftrightarrow$ (A5') for $\underline{W}^T$ is proven in [2, 21]. ■

For continuous operators $A : \mathcal{S}_1 \to \mathcal{S}_1$ we define $A_n = A^{\otimes n}$, $A_0 = 1$ and we set $\underline{A}^\otimes : \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ setting $\underline{A}^\otimes = \oplus_{n=0}^\infty A_n$. We get

Lemma A.2 *Let $A : \mathcal{S}_1 \to \mathcal{S}_1$ be linear and continuous. Then $\underline{W}^T \circ \underline{A}^\otimes = (\underline{W} \circ \underline{A}^\otimes)^T$ $\forall \underline{W} \in \underline{\mathcal{S}}'$.*

Since the scattering matrix can be considered as a bilinear functional on the Borchers' algebra, we require a definition of truncation for these objects. By the Schwartz kernel theorem it is clear that there is a one to one correspondence of the bilinear functionals $\underline{S}$ on $\underline{\mathcal{S}}$ with sets of tempered distributions $\{S_{n,m}\}_{n,m \in \mathbb{N}_0}$ where $S_{n,m} \in \mathcal{S}_{n+m}$ and $\underline{S}(\underline{f}, \underline{g}) = \sum_{n,m=0}^\infty S_{n,m}(f_n \otimes g_m)$. For $\lambda_l = (\lambda_l^1, \ldots, \lambda_l^r) \subseteq (1, \ldots, n), \nu_j = (\nu_j^1, \ldots, \nu_j^q) \subseteq (n+1, \ldots, n+m)$ we define $S(\lambda_l, \nu_j) = S_{r,q}(x_{\lambda_l^1}, \ldots, x_{\lambda_l^r}; x_{\nu_j^1}, \ldots, x_{\nu_j^q})$. With this notation we define recursively the truncated bilinear functional $\underline{S}^T$ associated with $\underline{S}$ via

$$S(1, \ldots, n; n+1, \ldots, n+m) = \sum_{\lambda \in \mathcal{P}(1, \ldots, n+m)} \prod_{l=1}^{|\lambda|} S^T(\lambda_l^<, \lambda_l^>). \tag{24}$$

Here $\lambda_l^< = \lambda_l \cap (1, \ldots, n)$ and $\lambda_l^> = \lambda_l \cap (n+1, \ldots, n+m)$.

The truncation of linear and bilinear functionals is related as follows: Let $\imath_\otimes$ be the injection of linear functionals into the bilinear functionals on $\underline{\mathcal{S}}$ given by $\imath_\otimes \underline{W}(\underline{f}, \underline{g}) = \underline{W}(\underline{f} \otimes \underline{g})$ $\forall \underline{f}, \underline{g} \in \underline{\mathcal{S}}$. Then we get from these definitions:

Lemma A.3 $\imath_\otimes \underline{W}^T = (\imath_\otimes \underline{W})^T$ $\forall \underline{W} \in \underline{\mathcal{S}}'$.

B Proof of Lemma 4.8 and Lemma 4.9

Proof of Lemma 4.8 We begin the proof of Lemma 4.8 with two auxiliary lemmas (for the definition of $\mathcal{S}_{1,2}, \mathcal{S}_{1,2}'$ cf. Sect. 4).

Lemma B.1 *The Fourier transform is a continuous mapping from $L^{1'}(\mathbb{R}, \mathbb{C})$ to $\mathcal{S}_{1,2}'(\mathbb{R}, \mathbb{C})$.*

Proof. We prove that $\mathcal{F} : \mathcal{S}_{1,2}(\mathbb{R}, \mathbb{C}) \to L^1(\mathbb{R}, \mathbb{C})$ is continuous. Then the statement of the lemma follows by duality. The stated continuity property is established by the following estimate:

$$\begin{aligned}
\|\mathcal{F}f\|_{L^1(\mathbb{R},\mathbb{C})} &= (2\pi)^{-1/2} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-i\xi t} f(\xi)\, d\xi \right| dt \\
&= (2\pi)^{-1/2} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-i\xi t} \left(1 - \frac{d^2}{d\xi^2}\right) f(\xi)\, d\xi \right| \frac{dt}{1+t^2} \\
&\leq \pi (2\pi)^{-1/2} \int_{\mathbb{R}} \left| \left(1 - \frac{d^2}{d\xi^2}\right) f(\xi) \right| d\xi \leq c \|f\|_{2,2},
\end{aligned} \tag{25}$$

for a sufficiently large constant $c > 0$ (here we have used $\int_{\mathbb{R}} dt/(1+t^2) = \pi$). ■

Let $1/\xi$ be defined as the Cauchy principal value of the function $1/\xi$ and the distribution $1/(\xi \pm i0)$ as the boundary value of $1/(\xi \pm i\epsilon)$ for $\epsilon \to +0$. $1/\xi$ and $1/(\xi \pm i0)$ are related via the Sokhotsky-Plemelji formula

$$\frac{1}{\xi \pm i0} = \frac{1}{\xi} \mp i\pi\delta(\xi) \ , \tag{26}$$

cf. [12] p. 45. These distributions can be understood as elements on $\mathcal{S}'_{1,2}(\mathbb{R}, \mathbb{C})$, since the Cauchy principle value is defined on $\mathcal{S}_{1,2}(\mathbb{R}, \mathbb{C})$ by [12] p. 44 and the delta distribution of course also is defined on this space. Furthermore, the Fourier transform (in $\mathcal{S}'_1(\mathbb{R}, \mathbb{C})$) of the step function $1_{\{0 \leq \pm s\}}$ is

$$\mathcal{F}_s(1_{\{0\leq \pm s\}}(s))(\xi) = (2\pi)^{-1/2} \frac{\mp i}{\xi \mp i0} \ , \tag{27}$$

see [12] p. 94.

Lemma B.2 $\lim_{t\to+\infty} e^{\pm i\xi t}/\xi = \pm i\pi\delta(\xi)$ *in* $\mathcal{S}'_{1,2}(\mathbb{R}, \mathbb{C})$.

Proof. We note that

$$\begin{aligned} \lim_{t\to+\infty} \frac{1}{\xi} e^{\pm i\xi t} &= \frac{1}{\xi} \lim_{t\to+\infty} \left[\int_0^t \frac{d}{ds} e^{\pm i\xi s} \ ds + 1 \right] \\ &= \pm i(2\pi)^{1/2} \lim_{t\to+\infty} \bar{\mathcal{F}}_s(1_{\{0\leq s\leq t\}})(\pm\xi) + \frac{1}{\xi}. \end{aligned}$$

Since by Lemma B.1 the (inverse) Fourier transform $\bar{\mathcal{F}}_s$ is continuous from $L^{1'}(\mathbb{R}, \mathbb{C})$ to $\mathcal{S}'_{1,2}(\mathbb{R}, \mathbb{C})$ and $1_{\{0\leq s\leq t\}}(s) \to 1_{\{0\leq s\leq \infty\}}(s)$ as $t \to +\infty$ in $L^{1'}(\mathbb{R}, \mathbb{C})$, we get for the r.h.s. of the above equation using also the formulae (26), (27):

$$\begin{aligned} \cdots &= \pm i(2\pi)^{1/2} \bar{\mathcal{F}}_s(1_{\{0\leq s\}})(\pm\xi) + \frac{1}{\xi} \\ &= \mp \left[\pm\frac{1}{\xi} - i\pi\delta(\xi) \right] + \frac{1}{\xi} = \pm i\pi\delta(\xi). \end{aligned}$$

■

Now we are in the position to prove Lemma 4.8. We only prove the lemma for a =out. The case a =in is in the same manner and the case a =loc is trivial.

We note that the function f in the expression $\chi_t(\text{out}, k)f(k)$, $f \in \mathcal{S}_{1,2}$, can be written as a sum of a function f_1 with supp $f_1 \subseteq \mathbb{R}^d_+ = (0,\infty) \times \mathbb{R}^{d-1}$ and supp $f_2 \subseteq \mathbb{R}^d_- = (-\infty, 0) \times \mathbb{R}^{d-1}$. Here we only deal with the "positive frequency part" f_1, and identify f_1 with the expression $\chi^+ f_1$, which does not change f_1 on the mass shell. Furthermore, we omit the index 1 in the following. Let thus $f \in \mathcal{S}_{1,2}$ with supp $f \subseteq \mathbb{R}^d_+$. Then

$$\begin{aligned} \lim_{t\to+\infty} \int_{\mathbb{R}^d} \frac{e^{i(k^0-\omega)t}}{k^2 - m^2} f(k) \ dk &= \lim_{t\to+\infty} \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} \frac{e^{i(k^0-\omega)t}}{k^2 - m^2} f(k) \ dk^0 \right] \ d\mathbf{k} \\ &= \lim_{t\to+\infty} \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} \frac{e^{i\xi t}}{\xi} \frac{f(\xi+\omega, \mathbf{k})}{\xi + 2\omega} \ d\xi \right] \ d\mathbf{k} \end{aligned}$$

where we have used the change of variables $k^0 \to \xi = k^0 - \omega$ in the last step. We note that $f(\xi+\omega,\mathbf{k})/(\xi+2\omega)$ is in $\mathcal{S}_{1,2}(\mathbb{R},\mathbb{C})$ for $\mathbf{k}\in\mathbb{R}^{d-1}$ since the denominator $(\xi+2\omega)$ is smooth on the support of $f(\xi+\omega,\mathbf{k})$. Thus, if we can interchange the $\int_{\mathbb{R}^{d-1}}\cdots d\mathbf{k}$ integral and the limit $\lim_{t\to+\infty}$ we get the formula of Lemma 4.8 by $\delta_m^+ = \delta(k^0-\omega)/2\omega$ and application of Lemma B.2.

Let $h\in\mathcal{S}_{1,2}(\mathbb{R}^d,\mathbb{C})$. We define $g_t(\mathbf{k}) = \int_{\mathbb{R}} \frac{e^{i\xi t}}{\xi} h(\xi,\mathbf{k})\,d\xi$. Using the product formula for the inverse Fourier transform on $\mathcal{S}'(\mathbb{R},\mathbb{C})$ we get

$$\begin{aligned}
|g_t(\mathbf{k})| &= 2\pi\left|\left[\bar{\mathcal{F}}_\xi(\frac{1}{\xi}) * \bar{\mathcal{F}}_\xi(h(\xi,\mathbf{k}))\right](t)\right| \\
&= 2\pi\left|\int_{\mathbb{R}} i(\pi - 1_{\{t-x>0\}}(t-x))\bar{\mathcal{F}}_\xi(h(\xi,\mathbf{k}))(x)\ dx\right| \\
&\le 2\pi(\pi+1)\int_{\mathbb{R}}\left|\bar{\mathcal{F}}_\xi(h(\xi,\mathbf{k}))(x)\right| dx \\
&\le c_1 \sup_{\xi\in\mathbb{R},0\le l\le 2}|(1+\xi^2)\frac{d^l}{d\xi^l}h(\xi,\mathbf{k})| \le c_2\frac{\|h\|_{2,d}}{(1+|\mathbf{k}|^2)^{d/2}},
\end{aligned}$$

for some $c_1,c_2>0$ sufficiently large. Here we made use of the estimate (25) and we also applied the formulae (26) and (27). But this estimate shows that there is an integrable majorant for $g_t, t\in\mathbb{R},$, namely $c/(1+|\mathbf{k}|^2)^{d/2}$, and we may therefore interchange the limit $t\to+\infty$ and the integral over $\mathbb{R}^{d-1}$ by the theorem of dominated convergence.

Proof of Lemma 4.9 For notational convenience we only prove the lemma for $j=1$. The proof for $j=2,\ldots,n-1$ can be carried out analogously. By integrating over the variables $k_2^0,\ldots,k_n^0$ and over $\mathbf{k}_2$ we obtain for the right hand side of (20)

$$\begin{aligned}
&\int_{\mathbb{R}^{(d-1)(n-2)}} \frac{f(k_1,(\omega_2,-\mathbf{k}_1-\sum_{l=3}^n\mathbf{k}_l),(\omega_3,\mathbf{k}_3),\ldots,(\omega_n,\mathbf{k}_n))}{\prod_{l=2}^n\omega_l} \\
\times\quad &\delta(k_1^0+\sum_{l=2}^n\omega_l)\ d\mathbf{k}_3\cdots d\mathbf{k}_n.
\end{aligned}$$

Here $\omega_2 = (|\mathbf{k}_1+\sum_{l=3}^n\mathbf{k}_l|^2+m^2)^{1/2}$. We set

$$h(k_1,\mathbf{k}_3,\ldots,\mathbf{k}_n) = \frac{f(k_1,(\omega_2,-\mathbf{k}_1-\sum_{l=3}^n\mathbf{k}_l),(\omega_3,\mathbf{k}_3),\ldots,(\omega_n,\mathbf{k}_n))}{\prod_{l=2}^n 2\omega_l}.$$

and we get that $h(k_1,\mathbf{k}_3,\ldots,\mathbf{k}_n)\in\mathcal{S}_{1,2}(\mathbb{R}^{d+(d-1)(n-2)},\mathbb{C})$ and $\|h\|_{2,L}\le c_L\|f\|_{2,L}$ for some $c_L>0$. We thus have to show, that for such h

$$g(k) = \int_{\mathbb{R}^{(d-1)(n-2)}} h(k,\mathbf{k}_3,\ldots,\mathbf{k}_n)\delta(\rho(\mathbf{k},\mathbf{k}_3,\ldots,\mathbf{k}_n)+k^0)\ d\mathbf{k}_3\cdots d\mathbf{k}_n \quad (28)$$

defines a $\mathcal{S}_{1,2}$-function g and that $\|g\|_{2,L}\le c_L\|h\|_{2,L'}$ for $c_L>0$ sufficiently large, where we have set $\rho(\mathbf{k},\mathbf{k}_3,\ldots,\mathbf{k}_n)=\sum_{l=2}^n\omega_l$.

Using a smooth partition of unity which has bounded derivatives we can write h as a sum of functions h_1, h_2, h_3 where on the support of h_1 we have $|\mathbf{k}| > 1$ and $|\sum_{l=3}^n \mathbf{k}_l| > 1$, on the support of h_2 we have $|\mathbf{k}| < 2, |\sum_{l=3}^n \mathbf{k}_l| > 1$ and on the support of h_3 we have $|\mathbf{k}| < 2, |\sum_{l=3}^n \mathbf{k}_l| < 2$. By the boundedness of derivatives of the partition of unity, $\|h_j\|_{2,L} \leq c_L \|f\|_{2,L}$ holds for $L \in \mathbb{N}_0, j = 1, 2, 3$ and sufficiently large $c_L > 0$. We denote the functions associated to h_j via Equation (28) by $g_j, j = 1, 2, 3$.

Let us first consider the right hand side of Equation (28) for h replaced by h_1 We introduce the variables $\mathbf{K}_j = \sum_{l=j}^n \mathbf{k}_l$ for $j = 3, \ldots, n$ and we set $\cos\theta_3 = \mathbf{K}_3 \cdot \mathbf{k}/(|\mathbf{K}_3||\mathbf{k}|)$ and $\cos\theta_j = \mathbf{K}_{j-1} \cdot \mathbf{K}_j/(|\mathbf{K}_{j-1}||\mathbf{K}_j|)$ for $j = 4, \ldots, n$. We then get

$$\begin{aligned}
\rho(\mathbf{k}, \mathbf{k}_3, \ldots, \mathbf{k}_n) &= \rho(\mathbf{k}, (|\mathbf{K}_3|, \cos\theta_3), \ldots, (|\mathbf{K}_n|, \cos\theta_n)) \\
&= (|\mathbf{k}|^2 + |\mathbf{K}_3|^2 + 2|\mathbf{k}||\mathbf{K}_3|\cos\theta_3 + m^2)^{1/2} \\
&+ \sum_{l=3}^{n-1} (|\mathbf{K}_l|^2 + |\mathbf{K}_{l+1}|^2 - 2|\mathbf{K}_l||\mathbf{K}_{l+1}|\cos\theta_{l+1} + m^2)^{1/2} \\
&+ (|\mathbf{K}_n|^2 + m^2)^{1/2}.
\end{aligned}$$

If we now change variables $\mathbf{k}_2, \ldots, \mathbf{k}_n \to \mathbf{K}_3, \ldots, \mathbf{K}_n$ in (28) and we then pass over to spherical coordinates $(|\mathbf{K}_l|, \cos\theta_l, \varpi_l), l = 3, \ldots, n$ where $(\cos\theta_l, \varpi_l)$ are coordinates on the sphere S^{d-2} (and the surface element on S^{d-2} is denoted by $d\cos\theta_l d\varpi_l$) we get

$$\begin{aligned}
&\int_{(S^{d-2}\times(0,\infty))^{\times n-2}} h_1(k, (|\mathbf{K}_3|, \cos\theta_3, \varpi_3), \ldots, (|\mathbf{K}_n|, \cos\theta_n, \varpi_n)) \\
\times\ & \delta(\rho(\mathbf{k}, (|\mathbf{K}_3|, \cos\theta_3), \ldots, (|\mathbf{K}_n|, \cos\theta_n)) + k^0) \\
\times\ & d\cos\theta_3 d\varpi_3 |\mathbf{K}_3|^{d-2} d|\mathbf{K}_3| \cdots d\cos\theta_n d\varpi_n |\mathbf{K}_n|^{d-2} d|\mathbf{K}_n| \qquad (29)
\end{aligned}$$

where we have written the function h_1 as a function of the new variables.

Using the formula

$$\delta(\rho(x) - a) = \sum_{y:\rho(y)=a} \frac{1}{|\rho'(y)|} \delta(x - y)$$

which holds if $\rho'(y) \neq 0$ if $\rho(y) = a$ and setting

$$\begin{aligned}
\varphi(\mathbf{k}, \mathbf{K}_3, \cos\theta_3) &= \frac{d}{d\cos\theta_3} \rho(\mathbf{k}, (|\mathbf{K}_3|, \cos\theta_3), \ldots, (|\mathbf{K}_n|, \cos\theta_n)) \\
&= \frac{|\mathbf{K}_3||\mathbf{k}|}{(|\mathbf{k}|^2 + |\mathbf{K}_3|^2 + 2|\mathbf{k}||\mathbf{K}_3|\cos\theta_3 + m^2)^{1/2}}
\end{aligned}$$

we get for (29)

$$\int_{(S^{d-2}\times(0,\infty))^{\times n-2}} h_1(k, (|\mathbf{K}_3|, \cos\theta_3, \varpi_3), \ldots, (|\mathbf{K}_n|, \cos\theta_n, \varpi_n))$$

$$
\begin{aligned}
\times & \quad \frac{\delta(\cos\theta_3 - \psi(k, |\mathbf{K}_3|, (|\mathbf{K}_4|, \cos\theta_4), \ldots, (|\mathbf{K}_n|, \cos\theta_n)))}{\varphi(\mathbf{k}, \mathbf{K}_3, \cos\theta_3)} \\
\times & \quad d\cos\theta_3 d\varpi_3 |\mathbf{K}_3|^{d-2} d|\mathbf{K}_3| \cdots d\cos\theta_n d\varpi_n |\mathbf{K}_n|^{d-2} d|\mathbf{K}_n| \qquad (30)
\end{aligned}
$$

where

$$
\begin{aligned}
& \psi(k, |\mathbf{K}_3|, (|\mathbf{K}_4|, \cos\theta_4), \ldots, (|\mathbf{K}_n|, \cos\theta_n)) \\
= & \Big[\Big(-k^0 - \sum_{l=4}^{n-1} (|\mathbf{K}_l|^2 + |\mathbf{K}_{l+1}|^2 - 2|\mathbf{K}_l||\mathbf{K}_{l+1}| \cos\theta_{l+1} + m^2)^{1/2} \\
- & (|\mathbf{K}_n|^2 + m^2)^{1/2} \Big)^2 - |\mathbf{k}|^2 - |\mathbf{K}_3|^2 - m^2 \Big] \Big/ (2|\mathbf{k}||\mathbf{K}_3|)
\end{aligned}
$$

is a smooth function on the set of arguments which are in the support of h_1. Furthermore, since $|\mathbf{k}|, |\mathbf{K}_3| > 1$ in the support of h_1, derivatives $(\partial^{|\alpha|}/\partial k^\alpha)\psi$ $(\partial^{|\alpha|}/\partial k^\alpha)\cos\theta$ also are bounded on the support of h_1 for any multiindex α.

We now set $\tilde{h}_1 = h_1/\varphi$ and we get that $\tilde{h}_1 \in \mathcal{S}_{1,2}(\mathbb{R}^{d+(d-1)(n-2)}, \mathbb{C})$ with $\|\tilde{h}_1\|_{2,L} \leq c_L \|h_1\|_{2,L}$ for some $c_L > 0$, $l \in \mathbb{N}_0$.

Consequently, we get for a multinindex α with $|\alpha| = 0, 1, 2$

$$
\begin{aligned}
& \left| \frac{\partial^{|\alpha|}}{\partial k^\alpha} \int_0^\infty \int_{-1}^1 \delta(\cos\theta_3 - \psi) \tilde{h}_1 \; d\cos\theta_3 |\mathbf{K}_3|^{d-2} d|\mathbf{K}_3| \right| \\
\leq & \quad c \|\tilde{h}_1\|_{2,L'} \frac{1}{(1+|k|^2)^{L/2}} \prod_{l=4}^n \frac{1}{(1+|\mathbf{K}_l|^2)^{d/2}},
\end{aligned}
$$

for c sufficiently large and $L' = \max\{L, d\}$. If we insert this estimate into (30), we get $\|g_1\|_{2,L} \leq c_L \|\tilde{h}_1\|_{2,L'} \leq c'_L \|h\|_{2,L'}$. If we can prove similar estimates for g_2, g_3, the proof is finished.

This is simple for g_2: We consider h_2 and g_2 as functions of the new variable $\mathbf{k}' = \mathbf{k} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^{d-1}$ with $|\mathbf{a}| \geq 3$. Then function h_2 the in these new variables fulfills the same conditions as h_1 before and we get the desired estimate.

It remains to show the estimate for g_3. Let $k, \mathbf{K}_3, \ldots, \mathbf{K}_n$ the coordinates introduced above. We define the vector field $\mathbf{b} = \mathbf{b}(\mathbf{K}_n) = 3\mathbf{K}_n/|\mathbf{K}_n|$ and we introduce new variables $\mathbf{k}' = \mathbf{k} - \mathbf{b}$, $\mathbf{K}'_l = \mathbf{K}_l + \mathbf{b}$, $l = 3, \ldots, n$. In the polar coordinates $|\mathbf{k}'|, |\mathbf{K}'_l|, \cos\theta'_3 = \mathbf{k}' \cdot \mathbf{K}'_3/(|\mathbf{k}'||\mathbf{K}'_3|), \cos\theta'_l = \mathbf{K}'_{l-1} \cdot \mathbf{K}'_l/(|\mathbf{K}'_{l-1}||\mathbf{K}'_l|)$ we then get for $\rho(k, \mathbf{k}_3, \ldots, \mathbf{k}_n)$:

$$
\begin{aligned}
& (|\mathbf{k}'|^2 + |\mathbf{K}'_3|^2 + 2|\mathbf{k}'||\mathbf{K}'_3| \cos\theta'_3 + m^2)^{1/2} \\
+ & \sum_{l=3}^{n-1} (|\mathbf{K}'_l|^2 + |\mathbf{K}'_{l+1}|^2 - 2|\mathbf{K}'_l||\mathbf{K}'_{l+1}| \cos\theta'_{l+1} + m^2)^{1/2} \\
+ & ((|\mathbf{K}'_n| + 3)^2 + m^2)^{1/2}
\end{aligned}
$$

and we can proceed as before, since $|\mathbf{k}'|, |\mathbf{K}'_3| > 1$ on the support of h_3.

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